ANALYSIS OF A TWO-DIMENSIONAL TEMPERATURE DISTRIBUTION IN A FINITE CYLINDER WITH NO INTERNAL HEAT SOURCES FOR BOUNDARY CONDITIONS OF THE FIRST KIND

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A nonstationary two-dimensional solution is presented for the temperature distribution in a finite cylinder whose surface temperature is a linear function of the time; the heating rates for the lateral and end surfaces are not the same. The solution is analyzed to refine the limits of applicability of the corresponding one-dimensional solutions for determining the diffusivity by a quasistationary method.

Quasistationary methods are the most effective way of investigating thermophysical characteristics over a wide range of temperatures [1-5]. Most of these methods, however, are constructed on one-dimensional solutions of the heat conduction equation. Boundaries of experimental specimens on which conditions are not specified theoretically distort the one-dimensional temperature distribution and lead to systematic errors which are difficult to allow for. A rigorous quantitative estimate of these errors necessitates solving the corresponding two- and three-dimensional heat conduction problems.

Volokhov [6] presents the solutions and analysis of two-dimensional temperature distributions in a finite cylinder for various combinations of boundary solutions of the first-third kinds which do not vary with time.

We analyze the nonstationary solution for a finite cylinder whose surface temperature is a linear function of the time; the heating rates of the lateral and end surfaces are different. The limits of applicability of the corresponding one-dimensional calculational formulas are refined, and a quantitative estimate is given of the possible systematic error in using finite cylinders and plates with different ratios of their linear dimensions and different ratios of the heating rates on their surfaces.

A finite cylinder of height 2h and diameter 2R with the origin of coordinates at its center is initially in thermal equilibrium with its surroundings; i.e., its temperature is equal to that of the surrounding medium T_0. At time t = 0 the lateral surface of the cylinder is heated at the constant rate of b_2 deg/sec and the end surfaces at the constant rate b_1. It is required to find the temperature distribution in the cylinder, i.e., to find a solution of the heat conduction equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) \tag{1}$$

which satisfies the initial and boundary conditions

$$\begin{gathered} T(r,\ z,\ 0) = T_0 = \text{const}, \\ T(r,\ h,\ \tau) = T_0 + b_1\tau, \\ T(R,\ z,\ \tau) = T_0 + b_2\tau, \\ \frac{\partial T(0,\ z,\ \tau)}{\partial r} = 0, \quad \frac{\partial T(r,\ 0,\ \tau)}{\partial z} = 0. \end{gathered} \tag{2}$$

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The solution of Eq. (1) obtained by using Hankel and Laplace transforms is

$$T-T_0=2b_1\tau\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_nJ_1(\mu_n)\operatorname{ch}\mu_nK}+b_2\tau\left(1-2\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_nJ_1(\mu_n)\operatorname{ch}\mu_nK}\right)$$

$$-\frac{b_2R^2}{4a}\left(1-\frac{r^2}{R^2}-8\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_n^3J_1(\mu_n)\operatorname{ch}\mu_nK}\right)+\frac{(b_1-b_2)Rz}{a}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{sh}\mu_n\frac{z}{R}}{\mu_n^2J_1(\mu_n)\operatorname{ch}\mu_nK}$$

$$-\frac{(b_1-b_2)hR}{a}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}\operatorname{sh}\mu_nK}{\mu_n^2J_1(\mu_n)\operatorname{ch}^2\mu_nK}+\frac{4h^2}{a}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(\lambda_m^2b_1+b_2\mu_n^2K)^2J_0\left(\mu_n\frac{r}{R}\right)\cos\lambda_m\frac{z}{h}}{\mu_nJ_1(\mu_n)\lambda_m(\lambda_m^2+\mu_n^2K^2)^2}$$

$$\times\exp\left[-(\lambda_m^2+\mu_n^2K^2)\right]\mathrm{Fo}_h, \tag{3}$$

where the μ_n are the positive roots of the characteristic equation

$$J_0(\mu)=0;$$

$$\lambda_m=(2m-1)\frac{\pi}{2};\ \mathrm{Fo}_h=\frac{a\tau}{h^2};$$

$K = h/R$ is the ratio of the length to the diameter of the cylinder.

Equation (3) can be written in the following dimensionless form:

$$\frac{\theta}{\mathrm{Pd}_h}=2\mathrm{Fo}_h\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_nJ_1(\mu_n)\operatorname{ch}\mu_nK}+C_b\mathrm{Fo}_h\left(1-2\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_nJ_1(\mu_n)\operatorname{ch}\mu_nK}\right)$$

$$-\frac{C_b}{4K^2}\left(1-\frac{r^2}{R^2}-8\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}}{\mu_n^3J_1(\mu_n)\operatorname{ch}\mu_nK}\right)+\frac{z}{h}\left(\frac{1-C_b}{K}\right)\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{sh}\mu_n\frac{z}{R}}{\mu_n^2J_1(\mu_n)\operatorname{ch}\mu_nK} \tag{4}$$

$$-\frac{1-C_b}{K}\sum_{n=1}^{\infty}\frac{J_0\left(\mu_n\frac{r}{R}\right)\operatorname{ch}\mu_n\frac{z}{R}\operatorname{sh}\mu_nK}{\mu_n^2J_1(\mu_n)\operatorname{ch}^2\mu_nK}+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}J_0\left(\mu_n\frac{r}{R}\right)(\lambda_m^2+C_b\mu_n^2K^2)\cos\lambda_m\frac{z}{h}}{\mu_nJ_1(\mu_n)\lambda_m(\lambda_m^2+\mu_n^2K^2)^2}$$

$$\times\exp\left[-(\lambda_m^2+\mu_n^2K^2)\right]\mathrm{Fo}_h,$$

where $\mathrm{Pd}_h = b_1h^2/aT_0$ is the Predvoditelev criterion; $c_b = b_2/b_1$; $\theta = t - T_0/T_0$.

The specimens used in practical thermophysical research ordinarily have the form of cylinders or plates. In the first case the radius of the cylinder R is the controlling dimension, and in the second case the thickness of the plate h; accordingly we consider the Fourier criteria $\mathrm{Fo}_R = a\tau/R^2$ and $\mathrm{Fo}_h = a\tau/h^2$. Solutions for an infinite cylinder ($b_1 = 0$, $h \to \infty$) and a plate ($b_2 = 0$, $R \to \infty$) follow from Eq. (3) as special cases. Depending on whether K is greater or less than unity it is expedient to use the following criteria:

$$\mathrm{Pd}_h=\frac{K^2}{C_b}\mathrm{Pd}_R,\ \mathrm{Fo}_h=\frac{1}{K^2}\mathrm{Fo}_R, \tag{5}$$

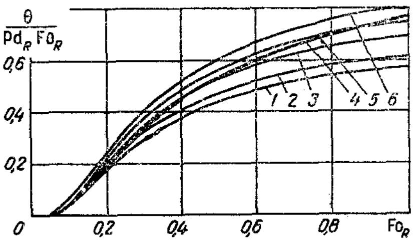
where

$$\mathrm{Pd}_R=\frac{b_2R^2}{aT_0}.$$

Solution (4), which is the same as (3), is simplified if $b_1 = b_2$. The structure of these solutions, however, makes the analysis difficult.

The generalized functions $\theta/\mathrm{Pd}_R\mathrm{Fo}_R$ and $\theta/\mathrm{Pd}_h\mathrm{Fo}_h$ at the center ($r = 0 = z$) are calculated on a Promin' computer for K = 1, 2, 3, 1/2, 1/3, and 1/4 for various values of $1/C_b$ and C_b.

Calculations of the two-dimensional temperature distributions $\theta/\mathrm{PdFo} = f(\mathrm{Fo})$ permit an estimate of the time of onset of the quasistationary thermal state, i.e., the time at which the nonstationary component (the double sum in the solutions) can be neglected in comparison with the quasistationary component. To



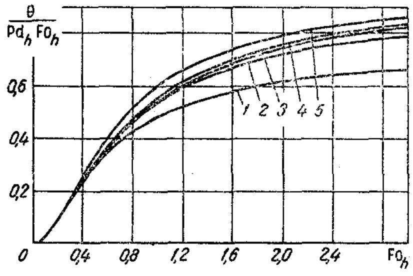


Fig. 2

Fig. 1. $\theta/Pd_R Fo_R$ at the center of a finite cylinder as a function of Fo_R for various values of K and $1/C_b$. 1) K = 1, $1/C_b = 0$; 2) 1 and 0.25, respectively; 3) 1 and 0.5; 4) center of infinite cylinder as Bi $\to \infty$, K = 3 and K = 2, $1/C_b = 0$ with a maximum relative error of 0.7 and 1.5%, respectively [1]; 5) 1 and 0.75; 6) 1 and 1.

Fig. 2. $\theta/Pd_h Fo_h$ at the center of a finite plate (disc) as a function of Fo_h for various values of K and C_b. 1) K = 1/2, $C_b = 0$; 2) 1/3 and 0, respectively; 3) center of infinite plate as Bi $\to \infty$, K = 1/4, $C_b = 0$ with a maximum relative error of 0.83%; 4) 1/3 and 1; 5) 1/2 and 1.

TABLE 1. Values of the Coefficients A_n, B_n, and D_n

K	$A_n = \sum_{n=1}^{\infty} \frac{1}{\mu_n J_1(\mu_n)\,\mathrm{ch}\,\mu_n K}$	$B_n = \sum_{n=1}^{\infty} \frac{1}{\mu_n^3 J_1(\mu_n)\,\mathrm{ch}\,\mu_n K}$	$D_n = \sum_{n=1}^{\infty} \frac{\mathrm{th}\,\mu_n K}{\mu_n^2 J_1(\mu_n)\,\mathrm{ch}\,\mu_n K}$
3	0,0012	0,0002	0,0005
2	0,0130	0,0023	0,0054
1	0,1394	0,0247	0,0579
1/2	0,3840	0,0743	0,1423
1/3	0,4699	0,0986	0,1415
1/4	0,4928	0,1096	0,1193

an accuracy of 1% the quasistationary regime begins for the values of K listed above and $C_b = 1$ at $Fo_R = 0.6$, 0.65, 0.7 and $Fo_h = 1.3$, 1.4, and 1.5, respectively.

Since the Fourier numbers are related by Eq. (5), to the indicated accuracy of the onset of the quasistationary state, it is not difficult to show that the corresponding times in studies with plates and cylinders of various dimensions but the same K can be the same or different.

Some of the results of the machine calculations of the generalized functions of Fourier numbers for various ratios of the heating rates and various values of K are shown in Fig. 1 and 2.

The maximum differences between the one- and two-dimensional generalized functions occur in the quasistationary thermal state. For a finite cylinder with K = 2 and K = 3 these differences under the most unfavorable experimental conditions ($1/C_b = 0$, $b_1 = 0$, $Fo_R \le 1$) are 1.5 and 0.7%, respectively; for $1/C_b = 1$, $b_1 = b_2$ they are 0.7 and 0.4%, respectively.

It should be emphasized that for equal heating rates the above differences in the quasistationary regime remain constant. For $b_1 \neq b_2$ these differences increase with time.

Figure 2 shows analogous relations for K < 1.

Maintaining constant heating rates on the surfaces of a finite cylinder in the steady state is equivalent to specifying constant heat fluxes across its boundaries. The corresponding expressions for the heat fluxes can be found from Eq. (3) by using Fourier's law. All conclusions on the character of the variation of the two-dimensional temperature distribution continue to hold.

In the linear heating of a uniform material with no internal heat sources it is common practice to determine the diffusivity. From the one-dimensional solutions for infinite cylinders and plates

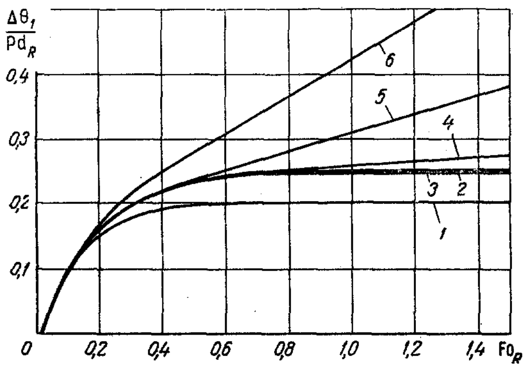


Fig. 3

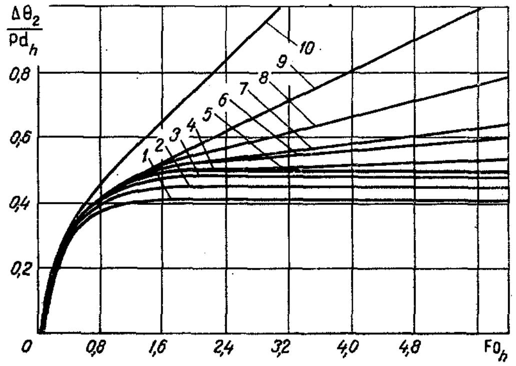


Fig. 4

Fig. 3. Relative drop $\Delta\theta_1/Pd_R$ as a function of Fo_R for various values of K and $1/C_b$: 1) K = 1, $1/C_b = 1$; 2) 2 and 1, respectively; 3) infinite cylinder, K = 3, $1/C_b = 0$ with a maximum relative error of 0.7%; 4) 2 and 0; 5) 1 and 0.5; 6) 1 and 0.

Fig. 4. Relative drop $\Delta\theta_2/Pd_h$ as a function of Fo_h for various values of K and C_b: 1) K = 1/2, $C_b = 1$; 2) 1/3 and 1, respectively; 3) 1/4 and 1; 4) infinite plate; 5) 1/4 and 0.5; 6) 1/4 and 0; 7) 1/3 and 0.5; 8) 1/3 and 0; 9) 1/2 and 0.5; 10) 1/2 and 0.

$$a = \frac{bR^2}{4\Delta T} = \frac{R^2}{4\Delta\tau}, \tag{6}$$

$$a = \frac{bh^2}{2\Delta T} = \frac{h^2}{2\Delta\tau}, \tag{7}$$

where ΔT is the temperature difference between the surface and the center of the specimens.

From Eq. (3) the expression for the diffusivity in a cylinder can be written in the form

$$a = \frac{b_2R^2}{4}\left[\frac{1-8B_n}{\Delta T_1 - 2(b_2-b_1)A_n\tau}\right] - \frac{(b_2-b_1)KR^2D_n}{\Delta T_1 - 2(b_2-b_1)A_n\tau}, \tag{8}$$

for a plate

$$a = \frac{b_1h^2}{2}\left\{\frac{C_b(1-8B_n)}{2K^2[\Delta T_2 - (b_1-b_2)(1-2A_n)\tau]}\right\} + \frac{(b_1-b_2)h^2D_n}{K[\Delta T_2 - (b_1-b_2)(1-2A_n)\tau]}, \tag{9}$$

where

$$\Delta T_1 = T(R,\ 0,\ \tau) - T(0,\ 0,\ \tau);\quad \Delta T_2 = T(0,\ h,\ \tau) - T(0,\ 0,\ \tau).$$

The values of A_n, B_n, and D_n are listed in Table 1.

If $b_1 = b_2 = b$, $\Delta T_1 = \Delta T_2 = \Delta T$. Then Eqs. (8) and (9) take the form

$$a = \frac{bR^2}{4\Delta T}(1-8B_n) = \frac{bh^2}{2\Delta T}\left(\frac{1-8B_n}{2K^2}\right), \tag{10}$$

for K = 3

$$a = \frac{bR^2}{\Delta T}\left(\frac{1}{4} - 0.0004\right) = 0.2496\,\frac{bR^2}{\Delta T}, \tag{11}$$

for K = 1/4

$$a = 0.4928\,\frac{bh^2}{\Delta T}. \tag{12}$$

Therefore Eqs. (11) and (12) agree with Eqs. (6) and (7) to an accuracy of 0.16 and 1.5%, respectively.

Other values of the coefficients for $b_1 = b_2$ appearing in Eq. (10) for various values of K can be found from Table 1. If $b_1 \neq b_2$ the calculation of the diffusivity at a definite time can be performed by Eqs. (8) and (9) using Table 1 and the fact that $\Delta T = f(\tau)$.

Figures 3 and 4 show $\Delta\theta_1/Pd_R$ as a function of Fo_R and $\Delta\theta_2/Pd_h$ as a function of Fo_h for various values of K, $1/C_b$, and C_b, and the corresponding one-dimensional relations. A knowledge and comparison of these relations permits the calculation of the relative error in determining the diffusivity by using Eqs. (6) and (7).

It should be noted that if the boundary condition $b_1 = b_2$ or a constant value of b_1/b_2 is not maintained in an experiment with plates or cylinders the corresponding values of C_b and $1/C_b$ will be functions of the temperature and time. Knowledge of T as a function of τ on the lateral surface of a plate or the end surfaces of a cylinder permits the determination of the instantaneous ratios C_b and $1/C_b$. These ratios will vary proportionally to the slope of the $T = f(\tau)$ graph at a given point.

As was shown above, the maximum deviations from the corresponding one-dimensional relations occur in the steady state. Consequently from the ratio C_b and $1/C_b$ at the end of the experiment it is possible to judge the percent deviation of the two-dimensional temperature distributions from the corresponding one-dimensional values, and in the last analysis the error in determining the diffusivity by using the one-dimensional calculation formulas.

Thus it is possible to find the limits of C_b and $1/C_b$ for a given K such that the error in determining the diffusivity by using the one-dimensional formulas does not exceed a certain value.

NOTATION

T	is the temperature at any point in a finite cylinder at any time, and T_0 is the initial temperature;
ΔT	is the difference in temperature between the surface and the center of the specimen;
$\Delta T_1 = T(R, 0, \tau) - T(0, 0, \tau)$;	
$\Delta T_2 = T(0, h, \tau) - T(0, 0, \tau)$;	
$\Delta\theta_1 = \Delta T_1/T_0$;	
$\Delta\theta_2 = \Delta T_2/T_0$;	
$\theta = T - T_0/T_0$;	
a	is the diffusivity;
τ	is the time;
R	is the radius of the cylinder;
h	is the half-height of the cylinder;
$K = h/R$;	
r and z	are running coordinates;
b_2 and b_1	are respectively the heating rates on the lateral and end surfaces;
$C_b = b_2/b_1$;	
$Pd_h = b_1h^2/aT_0$;	
$Pd_R = b_2R^2/aT_0$	are Predvoditelev numbers;
$Fo_h = a\tau/h^2$;	
$Fo_R = a\tau/R^2$	
J_0 and J_1	are zero and first order Bessel functions of the first kind;
μ_n	are the roots of $J_0(\mu) = 0$;
$\lambda_m = (2m-1)\pi/2$;	
A_n, B_n, and D_n	are constants for a given value of K.

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